# Stochastic stability of predator-prey model of Holling type(II) with term refuge

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# Introduction

General model describing the dynamic of predator-prey population

$$\begin{cases} \dfrac{dx}{dt} = xf(x) - g(x,y)y, \\ \dfrac{dy}{dt} = h(x,y)y. \end{cases}$$

$x\,(t)$ the density of prey population ;

$y\,(t)$ the density of predator population ;

$f(x)$ the per capita net prey growth in absence of predator ;
$h(x, y)$ the numerical response of predators (measures the growth rate of predators) ;
$g(x, y)$ the functional response of predator.

- Functional response : function giving the number of prey consumed by a predator per unit time.

Holling type II : describes a situation in which the number of prey consumed per predator initially rises quickly as the density of prey increases but then levels off with further increase in prey density.

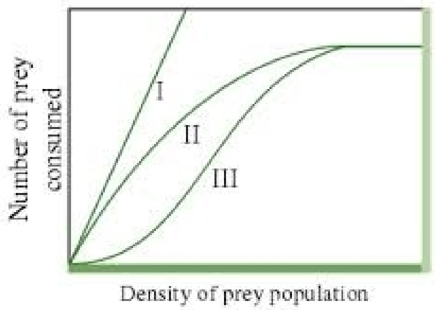


FIGURE : Three types of Holling functional response

# Deterministic model

In this work we will study the model of M. Alaoui, M.Daher but we treat the case where some preys using the refuge.
M. Aziz-Alaoui, M.Daher Okiye model (2003) :

$$\begin{cases} \dfrac{dx}{dt} = x(t)\left(r_1 - bx(t) - \dfrac{a_1 y(t)}{k_1 + x(t)}\right) \\ \dfrac{dy}{dt} = y(t)\left(r_2 - \dfrac{a_2 y(t)}{k_2 + x(t)}\right) \end{cases}$$

- Leslie and Gower in [3] and by Pielou in [4] :

$$\frac{dy}{dt} = ry(1 - \frac{y}{\alpha x})$$

$\frac{y}{\alpha x}$ : the loss in the predator population due to the rarity of its favorite food

- Leslie et Gower modified model :

$$\frac{dy}{dt} = ry(1 - \frac{y}{\alpha x + c})$$

# Position of the problem

We consider the system

$$(1)\qquad \begin{cases} \dfrac{dx}{dt} = x(t)\left(r_1 - bx(t) - \dfrac{a_1 y(t)}{k_1 + (x(t) - m)}\right) \\ \dfrac{dy}{dt} = y(t)\left(r_2 - \dfrac{a_2 y(t)}{k_2 + (x(t) - m)}\right) \end{cases},$$

with $x(0) \geqslant 0$ and $y(0) \geqslant 0$, where $x$(resp. $y$) : population density of preys (resp. population density of predators),
- $r_1$(resp. $r_2$) : growth rate of prey (resp. of predator),
- $b$ the strength of competition among individuals of species $x$,
- $a_1$(resp. $a_2$) : maximum value which per capita reduction rate of preys(resp. of predators),
- $k_1$(resp.$k_2$) : environment protection to prey(resp. to predator), and
- $m$ : number of preys using the refuge.

# Boundary of the model and existence of positively invariant attractor set

## Theorem 1

- *The set*

$$\mathbb{A} = \left\{(x, y) \in \mathbb{R}^2_+ : m \le x \le \frac{r_1}{b}, 0 \le x + y \le \mathbf{L}\right\},$$

  *where*
  $\mathbf{L} = \frac{1}{4a_2 b}\left(a_2 r_1 (r_1 + 4) + (r_2 + 1)^2 (r_1 + bk_2 - bm)\right).$
  *is positively invariant for solutions of the system* $(1)$,
- *All solutions of (1) initiating in* $\mathbb{R}^2_+$ *are ultimately bounded with respect to* $\mathbb{R}^2_+$ *and eventually enter the attracting set A.*

## Idea of the proof

To prove this theorem we need the following two lemmas.

### Lemma 1

*Positive quadrant* $\mathfrak{Int}\left(\mathbb{R}_+^2\right)$ *is invariant for system* $(1)$.

### Lemma 2

*Let* $\Phi$ *be an absolutely-continuous function satisfying the differential inequality*

$$\frac{d\Phi}{dt} + \alpha_1 \Phi(t) \leq \alpha_2, \qquad t \geq 0, \text{ where } (\alpha_1, \alpha_2) \in \mathbb{R}^2 \text{ avec } \alpha_1 \neq 0.$$

*Then*

$$\forall 0 \leq \tau \leq t, \quad \Phi(t) \leq \frac{\alpha_2}{\alpha_1} - \left(\frac{\alpha_2}{\alpha_1} - \Phi(\tau)\right) e^{-\alpha_1 (t-\tau)}.$$

# Idea of proof

1

- "$\mathbb{A}$ is positively invariant for solutions of system $(1)$" follows directly from Lemma 1.
  Indeed, as $(x(0), y(0)) \in \mathbb{A}$, $x(t)$ et $(y(t)$ remain positive.

# Idea of proof

1. - "$\mathbb{A}$ is positively invariant for solutions of system $(1)$" follows directly from Lemma 1.
   Indeed, as $(x(0), y(0)) \in \mathbb{A}$, $x(t)$ et $(y(t)$ remain positive.
2. "$\mathbb{A}$ is an attractor set" i.e.

$$\overline{\lim_{t\to+\infty}} x(t) \leq \frac{r_1}{b_1} \quad \text{et} \quad \overline{\lim_{t\to+\infty}} (x(t) + y(t)) \leq \mathbf{L}.$$

# Existence and uniqueness of the positive global solution

The system $(1)$ has three trivial equilibria (extinction of one or both populations)

$$E_0 = (0,0)\,,\ E_1 = \left(0, \frac{-r_2\left(-k_2+m\right)}{a_2}\right)\ et\ E_2 = \left(\frac{r_1}{b},0\right).$$

If we assume that we have

$$\frac{r_1\left(k_1-m\right)}{a_1} > \frac{r_2\left(k_2-m\right)}{a_2},$$

The system $(1)$ has a unique interior equilibrium $E^* = (x^*, y^*)$, where

$$\begin{aligned} x^* &= \frac{1}{2a_2 b}\left(a_2 r_1 + a_2 bm - a_1 r_2 - a_2 bk_1 + \sqrt{\Delta}\right), \\ y^* &= \frac{r_2\left(x^* + k_2 - m\right)}{a_2}, \end{aligned}$$

where

$$\Delta = \left(a_1 r_2 - a_2 r_1 + a_1 bk_1 - a_2 bm\right)^2 - 4a_2 b\left(a_1 r_2 k_2 - a_1 r_2 m - a_2 r_1 k_1 + a_2 r_1 m\right).$$

Suppose that $E^* = (x^*, y^*)$ is the equilibrium of the system $(1)$ ,

$$\begin{cases} r_1 - bx^* - \dfrac{a_1 y^*}{k_1 + (x^*(t) - m)} = 0 \\ r_2 - \dfrac{a_2 y^*}{k_2 + (x^*(t) - m)} = 0 \end{cases},$$

We get the equation of the second degree on $x^*$

$$a_2 b{x^*}^2 + (a_1 r_2 - a_2 r_1 + a_1 b k_1 - a_2 b m)\, x^* + a_1 r_2 k_2 - a_1 r_2 m$$

$$-a_2 r_1 k_1 + a_2 r_1 m = 0$$

If

$$\frac{r_1(k_1-m)}{a_1} > \frac{r_2(k_2-m)}{a_2}$$

is verified, than the system $(1)$ admits a unique interior equilibrium $E^* = (x^*, y^*)$ , given by

$$x^* = \frac{1}{2a_2 b}\left(a_2 r_1 + a_2 bm - a_1 r_2 - a_2 bk_1 + \sqrt{\Delta}\right),$$

$$y^* = \frac{r_2(x^* - m + k_2)}{a_2}.$$

# Global stability

In this section we prove global stability of the system $(1)$ by constructing a suitable Lyapunov function.

## Theorem 2

*The interior equilibrium $E^* = (x^*, y^*)$ is globally asymptotically stable if*

$$\mathbf{L} < \frac{r_1 (k_1 - m)}{2a_1},$$

$$m < k_1 < 2k_2, \qquad m < k_2$$

$$\textit{and} \;\; 4(r_1 + b(k_1 - m)) < a_1.$$

# Idée de la preuve

- The solutions of system $(1)$ are bounded and eventually enter the attractor set **A**, we can restrict the study to **A**.

Consider the positive Lyapunov function V defined by

$$\begin{aligned} V(x,y) &= V_1(x) + V_2(y) \text{ t.q.} \\ V_1(x) &= (x^* + k_1 - m)\left(x - x^* - x^* \ln\left(\frac{x}{x^*}\right)\right) \\ \text{et } V_2(y) &= \frac{a_1(x^* + k_2 - m)}{a_2}\left(y - y^* - y^* \ln\left(\frac{y}{y^*}\right)\right), \end{aligned}$$

# Idea of proof

$$\frac{dV}{dt} = \left(-b\left(x^* + k_1 - m\right) + \frac{a_1 y}{k_1 + x - m}\right)\left(x - x^*\right)^2$$

$$+ \left(-a_1 + \frac{a_1 y}{k_2 + x - m}\right)\left(x - x^*\right)\left(y - y^*\right) - a_1\left(y - y^*\right)^2 .$$

We can write $\frac{dV}{dt}$ in the following matrix form

$$\frac{dV}{dt} = -\left(x - x^*, y - y^*\right)\underbrace{\begin{pmatrix} -g(x,y) & -h(x,y) \\ -h(x,y) & a_1 \end{pmatrix}}_{M}\begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix},$$

# Idea of the proof

With

$$g(x,y) = -b\left(x^* + k_1 - m\right) + \frac{a_1 y}{k_1 + x - m}$$

and

$$h(x,y) = \frac{1}{2}\left(-a_1 + \frac{a_1 y}{k_2 + x - m}\right),$$

$\frac{dV}{dt} < 0 \Rightarrow M$ is positive definite.

(i) $g(x,y) < 0$

(ii) $\Phi(x,y) = -a_1 g(x,y) - h^2(x,y) < 0.$

- $\mathbf{L} < \frac{r_1(k_1-m)}{2a_1} \Rightarrow g(x,y) < 0$
- $$\left\{ \begin{array}{ll} m < k_1 < 2k_2, & m < k_2 \\ et\ \ 4\left(r_1 + b(k_1 - m)\right) < a_1 \end{array} \right. \Rightarrow (ii)$$

# Stochastic model

We consider the following stochastic system

$$(2) \quad \begin{cases} dx(t) = x(t)\left(r_1 - bx(t) - \dfrac{a_1 y(t)}{k_1 + (x(t) - m)}\right)dt + \sigma_1 (x - x^*)\, dW_1(t) \\ dy(t) = y(t)\left(r_2 - \dfrac{a_2 y(t)}{k_2 + (x(t) - m)}\right)dt + \sigma_2 (y - y^*)\, dW_2(t) \end{cases},$$

where $W_1(t)$ and $W_2(t)$ are two standard independent Wiener processes defined over the complete probability space $(\Omega, F, F_t, P)$
$\sigma_1, \sigma_2$ are real constants.

# Existence and uniqueness of the positive global solution

**Existence and uniqueness of the positive local solution**

## Lemma 3

*For any initial condition $(x_0, y_0) \in Int\mathbb{R}^2_+$, there is a unique positive local solution $(x(t), y(t))$ of the system (2) for $t \in [0.\tau_e)$ a.s where $\tau_e$ is the explosion time.*

# Proof

Using the transformation of variables
$K(t) = \ln(x - x^*)(t)$ and $L(t) = \ln(y - y^*)(t)$,

$$\begin{cases} dK(t) = (1 + \frac{x^*}{e^{K(t)}})\left(r_1 - b(e^{K(t)} + x^*) - \frac{a_1(e^{L(t)} + y^*)}{k_1 + e^{K(t)} + x^* - m} - \frac{\sigma_1^2}{2}\right) dt + \sigma_1 dW_1(t), \\ dL(t) = (1 + \frac{y^*}{e^{L(t)}})\left(r_2 - \frac{a_2 e^{L(t)} + y^*}{k_2 + e^{K(t)} + x^* - m} - \frac{\sigma_2^2}{2}\right) dt + \sigma_2 dW_2(t), \end{cases}$$

$K(0) = \ln(x(0) - x^*)$, $L(0) = \ln(y(0) - y^*)$.
There exist unique local solution $(K(t); L(t))$, for $t \in [0.\tau_e))$ where $\tau_e$ is the explosion time corresponding to the time when the solution may explode.
Hence $x(t) = e^{K(t)} + x^*$, $y(t) = e^{L(t)} + y^*$ is the unique positive of system (2).

# Existence and uniqueness of the positive global solution

Now we will show that this solution is global, i. e, it does not explode in finite which amounts to prove that $\tau_e = \infty$.

## Theorem 3

*For any initial condition* $(x_0, y_0) \in Int\mathbb{R}^2_+$*,there exists a unique solution* $(x(t), y(t)) \in Int\mathbb{R}^2_+$ *for the system* $(2)$*, for all* $\forall t \geq 0$ *a.s.*

# Proof

we can define the stopping time

$$\tau_r = \inf\left\{t \in [0, \tau_e) : x \notin (\tfrac{1}{r}, r) \text{ or } y \notin (\tfrac{1}{r}, r)\right\},$$

we have $\tau_\infty \leq \tau_e$ a.s.

To prove that $\tau_e = \infty$, it is sufficient to prove that $\tau_\infty = \infty$ a.s. let us assume the statement be false.

Let the function $V : Int\mathbb{R}_+^2 \to Int\mathbb{R}_+$ defined by

$$V(x, y) = (x + 1 - \log x) + (y + 1 - \log y).$$

Itô's formula and the positivity of $x(t)$ and $y(t)$ gives us

$$dV(x,y) \leq \Big[(r_1+b)x + (a_1+r_2+a_2)y + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}\Big]dt$$
$$+\sigma_1(x-1)dW_1 + \sigma_2(y-1)dW_2$$

For $t_1 \leq T$, the integration and expectation gives us
$(V(x_0,y_0)+c_5T)e^{c_5T} \geq E[1_{\Omega_r}(\omega)V(x(\tau_r,\omega),y(\tau_r,\omega))]$

$$\geq \varepsilon\Big[(r+1-\log r)\wedge(\frac{1}{r}+1+\log r)\Big],$$

where $1_{\Omega_r}$ is the indicator function of $\Omega_r$. Letting $r \to \infty$ we get$\infty > c_6 = \infty$ which leads us to a contradiction. So we must have $\tau_\infty = \infty$ a.s.

# Asymptotical stability

## Theorem 4

*The equilibrium of the system* $(2)$ *is stochastically asymptotically stable if*

$$\begin{cases} a_1 < \frac{2Ak_1}{\mathbf{L}-m}(2b - x^*\sigma_1^2), \\ \sigma_2^2 < \frac{2r_2}{a_1}, \\ k_2 > \mathbf{L} - m. \end{cases},$$

*where* $A = k_1 + x^* - m$, $B = k_2 + x^* - m$.

# Proof

We consider the positive Lyapunov function $V$ defined by

$$\begin{aligned}
V(x,y) &= V_1(x) + V_2(y) \text{ t.q.} \\
V_1(x) &= (x^* + k_1 - m)\left(x - x^* - x^* \ln\left(\frac{x}{x^*}\right)\right) \\
\text{et } V_2(y) &= \frac{a_1(x^* + k_2 - m)}{a_2}\left(y - y^* - y^* \ln\left(\frac{y}{y^*}\right)\right),
\end{aligned}$$

Applying the Itô's formula

$$\begin{aligned} dV &= dV_1 + dV_2 \\ &= LV + A\sigma_1(1 - \frac{x^*}{x})(x - x^*)dW_1 + \frac{Ba_1}{a_2}\sigma_2(1 - \frac{y^*}{y})(y - y^*)dW_2, \end{aligned}$$

where

$$\begin{aligned} LV &= (-Ab + \frac{ya_1}{k_1 + x - m} + \frac{Ax^*}{2x^2}\sigma_1^2)(x - x^*)^2 + -a_1(x - x^*)(y - y^*) \\ &- a_1(1 - \frac{Ba_1 y^* \sigma_2^2}{2a_2 y^2})(y - y^*)^2 + \frac{a_1 y}{k_2 + x - m}(y - y^*)(x - x^*). \end{aligned}$$

We have

$$
\begin{aligned}
LV \quad &\leq \quad (-Ab + \frac{a_1(\mathbf{L}-m)}{k_1} + \frac{Ax^*\sigma_1^2}{2})(x-x^*)^2 + (x-x^*)(y-y^*)(-a_1 + \frac{a_1}{k_2-m}) \\
&+ \quad a_1(\frac{B\sigma_2^2 y^*}{2a_2} - 1)(y-y^*)^2.
\end{aligned}
$$

In order that $LV < 0$, we should have

$$
\begin{cases}
a_1 < \frac{2Ak_1}{\mathbf{L}-m}(2b - x^*\sigma_1^2), \\
\sigma_2^2 < \frac{2r_2}{a_1}, \\
k_2 > \mathbf{L} - m.
\end{cases},
$$

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***Thank you for your attention***